PART : MATHEMATICS

1. Let y = f(x) is a solution of differential equation $e^y\left(\frac{dy}{dx}-1\right)=e^x$ and f(0) = 0 then f(1) is equal to :

(1) $\ell n2$ (2) $2+\ell n2$ (3) $1+\ell n2$ (4) $3+\ell n2$

Ans. **(3)**

Sol. $e^y = t$

$e^y\frac{dy}{dx} = \frac{dt}{dx}$

$\frac{dt}{dx} - t = e^x$

$IF = e^{\int -1.dx} = e^{-x}$

$t(e^{-x}) = \int e^x.e^{-x}dx$

$e^{y-x} = x + c$

Put x = 0, y = 0

$e^{y-x} = x + 1$

$y = x + \ell n(x + 1)$

at x = 1, $y = 1 + \ell n(2)$

2. If α is a roots of equation $x^2 + x + 1 = 0$ and $A = \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & \alpha & \alpha^2\\ 1 & \alpha^2 & \alpha\end{bmatrix}$ then A^{31} equal to :

(1) A (2) A^2 (3) A^3 (4) A^4

Ans. (3)

Sol. $A^2 = \frac{1}{3}\begin{bmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$

$\Rightarrow A^4 = I$

$\Rightarrow A^{30} = A^{28} \times A^3 = A^3$

3. The six digit numbers that can be formed using digits 1, 3, 5, 7, 9 such that each digit is used at least once.

Ans. **1800**

Sol. 1, 3, 5, 7, 9

For digit to repeat we have $^5C_1$ choice

And six digits can be arrange in $\frac{\lfloor 6}{\lfloor 2}$ ways.

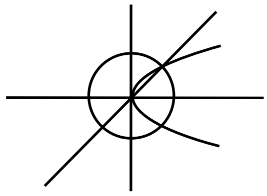
Hence total such numbers = $\frac{5\lfloor 6}{\lfloor 2}$

4. The area that is enclosed in the circle $x^2 + y^2 = 2$ which is not common area enclosed by $y = x$ & $y^2 = x$ is

(1) $\frac{1}{12}(24\pi - 1)$ (2) $\frac{1}{6}(12\pi - 1)$ (3) $\frac{1}{12}(6\pi - 1)$ (4) $\frac{1}{12}(12\pi - 1)$

Ans. **(2)**

Sol. Total area – enclosed area



$$2\pi - \int_0^1 \sqrt{x} - x\,dx$$

$$2\pi - \left(\frac{2x^{3/2}}{3} - \frac{x^2}{2}\right)_0^1$$

$$2\pi - \left(\frac{2}{3} - \frac{1}{2}\right) \Rightarrow 2\pi - \left(\frac{1}{6}\right) \Rightarrow \frac{12\pi - 1}{6}$$

5. If sum of all the coefficient of even powers in $(1 - x + x^2 - x^3 \ldots\ldots x^{2n})(1 + x + x^2 + x^3 \ldots\ldots + x^{2n})$ is 61 then n is equal to

(1) 30 (2) 32 (3) 28 (4) 36

Ans. **(1)**

Sol. Let $(1 - x + x^2 \ldots..)(1 + x + x^2 \ldots\ldots) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots$

put $x = 1$

$1(2n+1) = a_0 + a_1 + a_2 + \ldots\ldots a_{2n}$ ……..(i)

put $x = -1$

$(2n+1)\times 1 = a_0 - a_1 + a_2 + \ldots\ldots a_{2n}$ ……..(ii)

Form (i) + (ii)

$4n + 2 = 2(a_0 + a_2 + \ldots.)$

$= 2 \times 61$

$\Rightarrow 2n+1 = 61 \Rightarrow n = 30$

6. If variance of first n natural numbers is 10 and variance of first m even natural numbers is 16 then the value of m + n is

Ans. **18**

Sol. Var $(1, 2, \ldots\ldots, n) = 10 \Rightarrow \frac{1^2 + 2^2 + \ldots\ldots + n^2}{n} - \left(\frac{1 + 2 + \ldots\ldots + n}{n}\right)^2 = 10$

$$\Rightarrow \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = 10$$

$\Rightarrow n^2 - 1 = 120$ $\Rightarrow n = 11$

Var $(2, 4, 6, \ldots\ldots\ldots, 2m) = 16 \Rightarrow$ var $(1, 2, \ldots\ldots, m) = 4$

$\Rightarrow m^2 - 1 = 48 \Rightarrow m = 7 \Rightarrow m + n = 18$

7. Evaluate $\lim_{x\to 2} \dfrac{3^x + 3^{x-1} - 12}{3^{\frac{-x}{2}} - 3^{1-x}}$

Ans. **72**

Sol. Put $3^{\frac{x}{2}} = t$

$$\Rightarrow \lim_{t\to 3} \frac{\frac{4t^2}{3} - 12}{-\frac{3}{t^2} + \frac{1}{t}} = \lim_{t\to 3} \frac{4(t^2-9)t^2}{3(-3+t)} = \lim_{t\to 3} \frac{4t^2(3+t)}{3} = \frac{4\times 9\times 6}{3} = 72$$

8. If f(x) is continuous and differentiable in $x \in [-7, 0]$ and $f'(x) \le 2\ \forall x \in [-7, 0]$, also $f(-7) = -3$ then range of $f(-1) + f(0)$

(1) $[-5, -7]$ (2) $(-\infty, 6]$ (3) $(-\infty, 20]$ (4) $[-5, 3]$

Ans. **(3)**

Sol. Lets use LMVT for $x \in [-7, -1]$

$$\frac{f(-1) - f(-7)}{(-1+7)} \le 2$$

$$\frac{f(-1)+3}{6} \le 2 \Rightarrow f(-1) \le 9$$

Also use LMVT for $x \in [-7, 0]$

$$\frac{f(0) - f(-7)}{(0+7)} \le 2$$

$$\frac{f(0)+3}{7} \le 2 \Rightarrow f(0) \le 11 \qquad \therefore \qquad f(0) + f(-1) \le 20$$

9. If $y = mx + 4$ is common tangent to parabolas $y^2 = 4x$ and $x^2 = 2by$. Then value of b is

(1) –64 (2) –32 (3) –128 (4) 16

Ans. **(3)**

Sol. $y = mx + 4$ ……(i)

$y^2 = 4x$ tangent $\qquad y = mx + \dfrac{a}{m} \Rightarrow y = mx + \dfrac{1}{m}$ ………(ii)

from (i) and (ii)

$$4 = \frac{1}{m} \Rightarrow m = \frac{1}{4}$$

So line $y = \frac{1}{4}x + 4$ is also tangent to parabola $x^2 = 2by$, so solve

$$x^2 = 2b\left(\frac{x+16}{4}\right)$$

$\Rightarrow 2x^2 - bx - 16b = 0 \quad \Rightarrow D = 0 \Rightarrow b^2 - 4 \times 2 \times (-16b) = 0$

$\Rightarrow b^2 + 32 \times 4b = 0$

$b = -128$, $b = 0$ (not possible)

10. If α and β are the roots of equation (k+1) $\tan^2 x - \sqrt{2}\,\lambda$ tanx = 1 – k and $\tan^2(\alpha+\beta) = 50$. Find value of λ .

(1) 10 (2) 5 (3) 7 (4) 12

Ans. **(1)**

$(k+1)\tan^2 x - \sqrt{2}\lambda\ \tan x + (k-1)=0$

$\tan\alpha + \tan\beta = \dfrac{\sqrt{2}\lambda}{k+1}$

$\tan\alpha\,\tan\beta = \dfrac{k-1}{k+1}$

$\tan(\alpha+\beta)= (k-1)=0\,\dfrac{\dfrac{\sqrt{2}\,\lambda}{k+1}}{1-\dfrac{k-1}{k+1}} = \dfrac{\sqrt{2}\lambda}{2} = \dfrac{\lambda}{\sqrt{2}}$

$\tan^2(\alpha+\beta) = \dfrac{\lambda^2}{2} = 50$

$\lambda = 10$

11. Find image of point (2, 1, 6) in the plane containing points (2, 1, 0), (6, 3, 3) and (5, 2, 2)

(1) (6, 5, –2) (2) (6, –5, 2) (3) (2, –3, 4) (4) (2, –5, 6)

Ans. **(1)**

Sol. Plane is $x + y - 2z = 3$ $\Rightarrow$ $\dfrac{x-2}{1} = \dfrac{y-1}{1} = \dfrac{z-6}{-2} = \dfrac{-2(2+1-12-3)}{6} \Rightarrow (x, y, z) = (6, 5, -2)$

12. Let $(x)^k + (y)^k = (a)^k$ where $a, k > 0$ and $\dfrac{dy}{dx} + \left(\dfrac{y}{x}\right)^{\frac{1}{3}} = 0$, then find k

(1) $\dfrac{1}{3}$ (2) $\dfrac{2}{3}$ (3) $\dfrac{4}{3}$ (1) 2

Ans. **(2)**

Sol. $k.x^{k-1} + k.y^{k-1}\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\left(\dfrac{x}{y}\right)^{k-1}$

$\dfrac{dy}{dx} + \left(\dfrac{x}{y}\right)^{k-1} = 0$

$k - 1 = -\dfrac{1}{3}$

$k = 1 - \dfrac{1}{3} = \dfrac{2}{3}$

13. If $g(x) = x^2 + x - 1$ and $g(f(x)) = 4x^2 - 10x + 5$, then find $f\left(\frac{5}{4}\right)$.

(1) $\frac{1}{2}$ (2) $-\frac{1}{2}$ (3) $-\frac{1}{3}$ (4) $\frac{1}{3}$

Ans. (2)

Sol. $g(f(x)) = f^2(x) + f(x) - 1$

$$g\left(f\left(\frac{5}{4}\right)\right) = 4\left(\frac{5}{4}\right)^2 - 10.\ \frac{5}{4} + 5 = -\frac{5}{4}$$

$$g\left(f\left(\frac{5}{4}\right)\right) = f^2\left(\frac{5}{4}\right) + f\left(\frac{5}{4}\right) - 1$$

$$-\frac{5}{4} = f^2\left(\frac{5}{4}\right) + f\left(\frac{5}{4}\right) - 1$$

$$f^2\left(\frac{5}{4}\right) + f\left(\frac{5}{4}\right) + \frac{1}{4} = 0$$

$$\left(f\left(\frac{5}{4}\right) + \frac{1}{2}\right)^2 = 0$$

$$f\left(\frac{5}{4}\right) = -\frac{1}{2}$$

14. If $z = x + iy$ and real part $\left(\frac{z-1}{2z+i}\right) = 1$ then locus of z is

(1) Straight line with slope 2

(2) Straight line with slope $-\frac{1}{2}$

(3) circle with diameter $\frac{\sqrt{5}}{2}$

(4) circle with diameter $\frac{1}{2}$

Ans. (3)

Sol. $z = x + iy$

$$\left(\frac{z-1}{2z+i}\right) = \frac{(x-1)+iy}{2(x+iy)+i} = \frac{(x-1)+iy}{2x+(2y+1)i} \times \frac{2x-(2y+1)i}{2x-(2y+1)i}$$

$$\text{Re}\left(\frac{z+1}{2z+i}\right) = \frac{2x(x-1)+y(2y+1)}{(2x)^2+(2y+1)^2} = 1$$

$\Rightarrow 2x^2 + 2y^2 - 2x + y = 4x^2 + 4y^2 + 4y + 1$ $\quad\Rightarrow 2x^2 + 2y^2 + 2x + 3y + 1 = 0$

$\Rightarrow x^2 + y^2 + x + \dfrac{3}{2}y + \dfrac{1}{2} = 0$ Circle with centre $\left(-\dfrac{1}{2} - \dfrac{3}{4}\right)$

$r = \sqrt{\dfrac{1}{4} + \dfrac{9}{16} - \dfrac{1}{2}} = \sqrt{\dfrac{4+9-8}{16}} = \dfrac{\sqrt{5}}{4}$

15. If distance between the foci of an ellipse is 6 and distance between its directrices is 12, then length of its latus rectum is

(1) 4 (2) $3\sqrt{2}$ (3) 9 (4) $2\sqrt{2}$

Ans. (2)

Sol. $2ae = 6$ and और $\dfrac{2a}{e} = 12$

$\Rightarrow ae = 3$ and और $\dfrac{a}{e} = 6$

$\Rightarrow a^2 = 18$

$\Rightarrow b^2 = a^2 - a^2e^2 = 18 - 9 = 9$ $\quad\Rightarrow$ L.R. $= \dfrac{2b^2}{a} = \dfrac{2\times 9}{3\sqrt{2}} = 3\sqrt{2}$

16. If $y = \sqrt{\dfrac{2(\tan\alpha + \cot\alpha)}{1+\tan^2\alpha} + \dfrac{1}{\sin^2\alpha}}$ when $\alpha \in \left(\dfrac{3\pi}{4}, \pi\right)$ then find $\dfrac{dy}{d\alpha}$ at $\alpha = \dfrac{5\pi}{6}$

(1) 4 (2) 2 (3) 3 (4) –4

Ans. (1)

Sol. $y = \sqrt{\dfrac{2\cos^2\alpha}{\sin\alpha\cos\alpha} + \dfrac{1}{\sin^2\alpha}} = \sqrt{2\cot\alpha + \text{cosec}^2\alpha}$ $= |1 + \cot\alpha| = -1 - \cot\alpha$

$\dfrac{dy}{d\alpha} = \text{cosec}^2\alpha \Rightarrow \left(\dfrac{dy}{d\alpha}\right) \text{at } \alpha = \dfrac{5\pi}{6} \text{ will be} = 4$

17. If A(1, 1), B(6, 5) , $C\left(\frac{3}{2},2\right)$ are vertices of $\triangle$ABC. A point P is such that area of $\triangle$PAB, $\triangle$PAC, $\triangle$PBC are equal, also $Q\left(\frac{-7}{6},\frac{-1}{3}\right)$, then length of PQ is

(1) 2 (2) 3 (3) 4 (4) 5

Ans. **(4)**

Sol. P will be centroid of $\triangle$ABC

$P\left(\frac{17}{6},\frac{8}{3}\right) \quad \Rightarrow \quad PQ = \sqrt{\left(\frac{24}{6}\right)^2 + \left(\frac{9}{3}\right)} = 5$

18. $(p \to q) \wedge (q \to \sim p)$ is equivalent to

(1) ~p (2) p (3) $p \wedge q$ (4) $p \vee q$

Ans. **(1)**

Sol.

p	**q**	**p → q**	**~P**	**q → ~p**	**(p → q) ∧ (p → ~q)**
T	T	T	F	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	T	T

Clearly $(p \to q) \wedge (q \to \sim p)$ is equivalent to ~p

19. Find greatest value of k for which $49^k + 1$ is factor of $1 + 49 + 49^2 \ldots..(49)^{125}$

(1) 63 (2) 65 (3) 2 (4) 5

Ans. **(1)**

Sol. $\frac{(49)^{126} - 1}{48} = \frac{((49)^{63} + 1)(49^{63} - 1)}{48}$

20. If f(x) = |2 – |x – 3|| is non differentiable in x $\in$ S.Then value of $\sum_{x \in S} (f(f(x))$ is

Ans. 3

Sol. $\because$ f(x) is non differentiable at x = 1, 3, 5

$\Sigma f(f(x)) = f(f(1) + f(f(3)) + f(f(5))$

$= 1 + 1 + 1$

$= 3$

21. If system of equations

$2x + 2ay + az = 0$

$2x + 3by + bz = 0$

$2x + 4cy + cz = 0$ have non-trivial solution

then

(1) $a + b + c = 0$

(2) a,b,c are in A.P.

(3) $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ are in A.P.

(4) a, b, c in G.P.

Sol. For non-trivial solution

$$\begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$$

$(3bc - 4bc) - (2ac - 4ac) + (2ab - 3ab) = 0$

$-bc + 2ac - ab = 0$

$ab + bc = 2ac$

a, b, c in H.P.

$\Rightarrow \quad \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ in A.P.

22. If sum of 5 consecutive terms of 'an A.P is 25 & product of these terms is 2520. If one of the terms is – 1/2 then the value of greatest term is

(1) $\frac{21}{2}$ (2) 16 (3) 5 (4) 7

Ans. **(2)**

Sol. Let terms be (माना कि पद) $a - 2d, a - d, a, a + d, a + 2d$.

sum योगफल= 25 $\Rightarrow$ $5a = 25 \Rightarrow a = 5$

Product गुणनफल= 2520

$(5-2d)(5-d)5(5+d)(5+2d) = 2520$

$\Rightarrow (25-4d^2)(25-d^2) = 504$

$\Rightarrow 625 - 100d^2 - 25d^2 + 4d^4 = 504$

$\Rightarrow 4d^4 - 125d^2 + 625 - 504 = 0$

$\Rightarrow 4d^4 - 125d^2 + 121 = 0$

$\Rightarrow 4d^4 - 121d^2 - 4d^2 + 121 = 0$

$\Rightarrow (d^2 - 1)(4d^2 - 121) = 0$

$\Rightarrow d = \pm 1, \quad d = \pm \frac{11}{2}$

$d = \pm 1$, does not give $\frac{-1}{2}$ as a term

$\therefore d = \frac{11}{2}$

$\therefore$ Largest term $= 5 + 2d = 5 + 11 = 16$

23. Let $\vec{a} = \alpha\hat{i} + 2\hat{j} + \beta\hat{k}$

$\vec{a}$ lies in plane of $\vec{b} \& \vec{c}$

$\vec{b} = \hat{i} + \hat{j}$ & $\vec{c} = \hat{i} - \hat{j} + 4\hat{k}$

of $\vec{a}$ bisectors angle between $\vec{b}$ & $\vec{c}$, then

(1) $\vec{a} \cdot \hat{k} + 2 = 0$ (2) $\vec{a} \cdot \hat{k} + 4 = 0$ (3) $\vec{a} \cdot \hat{k} - 2 = 0$ (4) $\vec{a} \cdot \hat{k} + 5 = 0$

Ans. **(1)**

Sol. angle bisector can be $\vec{a} =$ $\vec{a} = \lambda(\hat{b} + \hat{c})$ or $\vec{a} = \mu(\hat{b} - \hat{c})$

$$\vec{a} = \lambda\left(\frac{\hat{i} + \hat{j}}{\sqrt{2}} + \frac{\hat{i} + \hat{j} + 4\hat{k}}{3\sqrt{2}}\right) = \frac{\lambda}{3\sqrt{2}}\left[3\hat{i} + 3\hat{j} + \hat{i} - \hat{j} + 4\hat{k}\right] = \frac{\lambda}{3\sqrt{2}}\left[4\hat{i} + 2\hat{j} + 4\hat{k}\right]$$

Compare with $\vec{a} = \alpha\hat{i} + 2\hat{j} + \beta\hat{k}$

$$\frac{2\lambda}{3\sqrt{2}} = 2 \Rightarrow \lambda = 3\sqrt{2}$$

$$\vec{a} = 4\hat{i} + 2\hat{j} + 4\hat{k}$$

Not in option so now consider $\vec{a} = \mu\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}} - \frac{\hat{i}-\hat{j}+4\hat{k}}{3\sqrt{2}}\right)$

$$\vec{a} = \frac{\mu}{3\sqrt{2}}\left(3\hat{i} + 3\hat{j} - \hat{i} + \hat{j} - 4\hat{k}\right)$$

$$= \frac{\mu}{3\sqrt{2}}\left(2\hat{i} + 4\hat{j} - 4\hat{k}\right)$$

Compare with $\vec{a} = \alpha\hat{i} + 2\hat{j} + \beta\hat{k}$

$$\frac{4\mu}{3\sqrt{2}} = 2 \Rightarrow \mu = \frac{3\sqrt{2}}{2}$$

$$\vec{a} = \hat{i} + 2\hat{j} - 2\hat{k}$$

$$\vec{a}.\hat{k} + 2 = 0$$

$$-2 + 2 = 0$$

24. Given f(a + b + 1 – x) = f(x) $\forall$ x $\in$ R then the value of $\frac{1}{(a+b)}\int_a^b x[f(x)+f(x+1)]dx$ is equal to

(1) $\int_{a+1}^{b+1} f(x)\,dx$ (2) $\int_{a+1}^{b+1} f(x+1)\,dx$ (3) $\int_{a-1}^{b-1} f(x)\,dx$ (4*) $\int_{a-1}^{b-1} f(x+1)\,dx$

Ans. **(4)**

Sol. $I = \frac{1}{(a+b)}\int_a^b x[f(x)+f(x+1)]dx$ ......(1)

$x \to a + b - x$

$$I = \frac{1}{(a+b)}\int_a^b (a+b-x)[f(a+b-x)+f(a+b+1-x)]dx$$

$I = \frac{1}{(a+b)}\int_a^b (a+b-x)[f(x+1)+f(x)]dx$ ....(2)

[$\because$ put x $\to$ x + 1 in given equation]

(1) + (2)

$$2I = \int_a^b [f(x+1)+f(x)]dx$$